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Debye Screening for Two-Dimensional Coulomb Systems at High Temperatures

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The grand canonical ensemble of a two-dimensional Coulomb system with ± 1 charges is proved to have screening phenomena in its high-temperature region. The Coulomb potential in a finite region Λ is assumed to be $(-\Delta_\Lambda)^{-1}$, where Δ_Λ is the Laplacian with zero boundary conditions on Λ. The hard-core condition is not assumed. The model is set up by separating $(-\Delta_\Lambda)^{-1}$ into a short-range part and a long-range part depending on a parameter λ. The self-energies are subtracted only for the short-range part and therefore a choice of λ is a choice of subtraction of self-energies. The method of proof is in general the same as that of Brydges–Federbush "Debye screening," except that here a modification for the short-range part of the potentials is needed.

KEY WORDS: Sine-Gordon field; Coulomb systems; Debye screening; cluster expansion; Mayer's expansion; decay of correlation functions.

1. INTRODUCTION

1.1. Brydges and Federbush[(2)] have proved that screening phenomena occur for a clssical Coulomb system in three dimensions. They considered systems of s species of particles. For simplicity, we describe their results for two species of particles with charges $e = \pm 1$. Let $\Lambda \subset \Lambda'$ be rectangular regions in R^3. Let Δ_Λ be the Laplacian Δ with zero boundary conditions on Λ. Let

$$u(x, y) = (-\Delta_\Lambda)^{-1}(x, y) - (-\Delta_\Lambda + \lambda^{-2} l_{\mathrm{D}}^{-2})^{-1}(x, y)$$

Let

$$v_{2,e_i e_j}(x_i, x_j) = e_i e_j(-\Delta_{\Lambda'} + \lambda^{-2} l_{\mathrm{D}}^{-2})^{-1}(x_i, x_j) + w_{e_i, e_j}(x_i, x_j)$$

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In Ref. 2 the partition function in Λ' is defined by

$$Z_{\Lambda,\Lambda'} = \sum_{n=0}^{\infty} \frac{\tilde{z}^n}{n!} \int_{\Lambda'} dx_1 \cdots \int_{\Lambda'} dx_n \sum_{e_1,\ldots,e_n} e^{-\beta U} e^{-\beta W}$$

Here $e_i = \pm 1$ for all $i = 1, 2,\ldots,$ and

$$U = \tfrac{1}{2} \sum_{1 \leqslant i,j \leqslant n} e_i e_j u(x_i, x_j), \qquad W = \tfrac{1}{2} \sum_{1 \leqslant i \neq j \leqslant n} v_{2,e_ie_j}(x_i, x_j)$$

They considered the limit of the system when $\Lambda' \nearrow R^3$ and $\Lambda \nearrow R^3$. They proved the existence and exponential clustering of correlation functions of charge densities at high temperature under some conditions. One of the conditions is as follows. v_2 is assumed to be decomposed as $w_N + w_R$, where $w_R \geqslant 0$, and there exists a constant B such that $\sum_{1 \leqslant i \neq j \leqslant n} w_N(x_i, x_j) \geqslant -Bn$, for all n.

1.2. We set up our two-dimensional Coulomb system by replacing R^3 by R^2, $\Lambda = \Lambda'$, and we put $w_{e_ie_j} = 0$. We prove, when $\Lambda \nearrow R^2$, the existence and exponential clustering of correlation functions of charge densities for all sufficiently small positive numbers λ and all sufficiently small β depending on λ.

If we put

$$z = \tilde{z} \exp \beta \left\{ -u(0,0) + \frac{1}{4\pi} \log \frac{|\Lambda|}{\lambda^2 l_{\mathrm{D}}^2 \pi} \right\}$$

where $|\Lambda|$ is the area of Λ, in Section 2.2, we shall relate our system to the system defined by the infinite-volume limit of the system in Λ with the partition function

$$Z'_\Lambda = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda} dx_1 \cdots \int_{\Lambda} dx_n \sum_{e_1,\ldots,e_n} e^{-\beta H}$$

where

$$\begin{aligned} H &= \sum_{1 \leqslant i < j \leqslant n} (e_i e_j / 2\pi) \log(\lambda l_{\mathrm{D}} / |x_i - x_j|) \qquad &&\text{if } \sum_{i=1}^{n} e_i = 0 \\ &= \infty &&\text{if } \sum_{i=1}^{n} e_i \neq 0 \end{aligned}$$

This system, with $\lambda = 1$, has been considered in Ref. 5. The λ in Z'_Λ can be changed to 1 by redefining the activity z to be $z \exp(-\log \lambda / 4\pi)$, so our model differs from the model in Ref. 5 in the boundary conditions (which are Dirichlet instead of free) on the Coulomb potential. At present, the problem with free boundary conditions is much harder. For a three-dimensional Coulomb system, free boundary conditions have been considered in Ref. 4.

1.3. Since we set $w_{e_ie_j}=0$, the short-range potential v_2 in our case no longer satisfies the condition described in 1.1. We cannot use the same criterion as the one in Ref. 2 for the convergence of Mayer's expansion for the short-range potentials. Instead, we use an "iterated Mayer expansion" and a criterion for its convergence from Ref. 3 to deal with our short-range potentials. The rest of our proofs, including the long-range potentials and the combination of the two parts of potentials, are based on the same arguments as those in Ref. 2. Our development is thus parallel with that of Ref. 2. We assume reader is familiar with the proofs in Ref. 2.

2. DEFINITIONS OF THE SYSTEM AND THE MAIN RESULT

2.1. Let Λ be a domain in R^2. Let Δ be the Laplacian in R^2 and Δ_Λ be Δ with zero boundary conditions in Λ. For any $\lambda > 0$, we define

$$u(x, y)=(-\Delta_\Lambda)^{-1}(x, y)-(-\Delta_\Lambda+\lambda^{-2}l_{\mathrm{D}}^{-2})^{-1}(x, y)$$

$$w(x, y)=(-\Delta_\Lambda+\lambda^{-2}l_{\mathrm{D}}^{-2})^{-1}(x, y)$$

For $\Lambda \subset R^2$, we consider the grand canonical ensemble of particles with charges $+1$ or -1 defined by the partition function Z_Λ,

$$Z_\Lambda=\sum_{n=0}^{\infty}\frac{\tilde{z}^n}{n!}\int_\Lambda dx_1\cdots\int_\Lambda dx_n \sum_{e_1\cdots e_n} e^{-\beta V_n}$$

Here $e_i=\pm 1$, $V_n=U+W$, and

$$U=\tfrac{1}{2}\sum_{1\leqslant i,j\leqslant n} e_ie_ju(x_i, x_j), \qquad W=\tfrac{1}{2}\sum_{1\leqslant i\neq j\leqslant n} e_ie_jw(x_i, x_j)$$

We choose $l_{\mathrm{D}}=(2\tilde{z}\beta)^{-1/2}$; $l_{\mathrm{D}}\geqslant 0$ is called the Debye length. We note that our system can be fit into the framework of Ref. 2 if we replace v_2 in Ref. 2 by our $e_ie_jw(x_i, x_j)$. (In Ref. 2, two boxes Λ and Λ' are considered. In our situation, because of charge symmetry, it is sufficient to consider only one box Λ.)

Let $\sigma_e(x) = \sum_i \delta_x(x_i)\,\delta_e(e_i)$ be the density of charge e at x. Let $J(x) = \sum_e e\sigma_e(x)$ be the total charge density at point x. If A is a functional of σ_e, we shall write

$$I(A) = \sum_{n=0}^{\infty} \frac{\tilde{z}^n}{n!} \int e^{-\beta V_n} A$$

$$\langle A \rangle_\Lambda = I(A)/I(1), \qquad Z = I(1)/Z_0$$

where $Z_0 = I(1)$ calculated with u set to 0. We shall obtain the infinite-volume limit $\langle A \rangle$ by letting $\Lambda \nearrow R^2$.

2.2. Let $|\Lambda|$ be the area of Λ. If we put

$$z = \tilde{z} \exp \beta\{-u(0, 0) + (1/4\pi) \log(|\Lambda|/\lambda^2 l_D^2 \pi)\}$$

then

$$Z_\Lambda = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_n \sum_{e_1 \cdots e_n} e^{-\beta H(\Lambda)}$$

where

$$H(\Lambda) = (1/2)\Bigg\{\sum_{i \neq j} e_i e_j [u(x_i, x_j) - (1/4\pi) \log(|\Lambda|/\lambda^2 l_D^2 \pi) + w(x_i, x_j)] + (1/4\pi) \log(|\Lambda|/\lambda^2 l_D^2) \Big(\sum e_i\Big)^2 + \sum_i [u(x_i, x_i) - u(0, 0)]\Bigg\}$$

If we take Λ to be the disc with radius $\lambda l_D l$ and center at origin, then

$$(-\Delta_\Lambda)^{-1}(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} - \frac{1}{2\pi} \log \frac{1}{\lambda l_D l + |x|} - h_l(x, y) \tag{2.1}$$

where, for all x, $h_l(x, y)$ is the harmonic function of $y \in \Lambda$ such that $(-\Delta_\Lambda)^{-1}(x, y) = 0$, for all $y \in \partial\Lambda$. By the maximum principle of harmonic functions,

$$|h_l(x, y)| \leqslant \frac{1}{2\pi} \log \frac{1}{\lambda l_D l - |x|} - \frac{1}{2\pi} \log \frac{1}{\lambda l_D l + |x|} \tag{2.2}$$

We shall also use the following facts:

$$\lim_{x \to y} \left[(-\Delta_\Lambda + \lambda^{-2} l_D^{-2})^{-1}(x, y) - \frac{1}{2\pi} \log \frac{\lambda l_D}{|x-y|} \right] = L_\Lambda(y), \tag{2.3}$$

where $L_\Lambda(y)$ is finite for all $y \in \Lambda$,

$$\lim L_\Lambda(y) = \frac{1}{2\pi}(\log 2 - \gamma) \qquad \text{as} \quad \Lambda \nearrow R^2$$

and γ is Euler's constant.

By (2.1)–(2.3), we can prove that

$$\lim_{l \to \infty} [u(x, x) - u(0, 0)] = 0$$

$$\lim_{l \to \infty} \left[u(x, x) - \frac{1}{4\pi} \log \frac{|\Lambda|}{\lambda^2 l_D^2 \pi} \right] = \frac{1}{2\pi}(\log 2 - \gamma)$$

$$\lim_{l \to \infty} H(\Lambda) = \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{2\pi} \log \frac{\lambda l_D}{|x_i - x_j|} \qquad \text{if} \quad \sum_i e_i = 0$$

$$= \infty \qquad \text{if} \quad \sum_i e_i \neq 0$$

Intuitively, these mean that the infinite-volume limit of our system is a description of a neutral system, i.e., $\sum e_i = 0$, with pair interacting potential $(1/2\pi)\log(\lambda l_D/|x - y|)$ (see, e.g., Ref. 5). This system depends on λ. The appearance of λ may be explained as follows. The two-dimensional Coulomb system is parametrized by an inverse temperature β that is dimensionless and an activity z with dimension length^{-2}. If we use the Green functions of the Laplacian in R^2 to define the Coulomb potential, an ambiguity arises. There is a one-parameter family $(2\pi)^{-1} \log L/|x - y|$ of Green functions, where L is a length. A choice of L will set a length scale so that zL^2 is a dimensionless measure of the density. The choice of L in the Green function amounts to a choice of how to subtract self-energies in the free boundary conditions case because

$$\sum_{i<j} e_i e_j (2\pi)^{-1} \log(L/|x_i - x_j|)$$
$$= \sum_{i<j} e_i e_j (2\pi)^{-1} \log(1/|x_i - x_j|) + \sum_{i<j} e_i e_j (2\pi)^{-1} \log L$$

and since $\sum e_i = 0$, the second term equals $-2^{-1} n (2\pi)^{-1} \log L$, which is proportional to n and therefore it is a self-energy (or equivalently a redefinition of z). This ambiguity in the Green function surfaces as an ambiguity in how to define and subtract self-energy in the Dirichlet case. If the self-energies are omitted, the partition function is divergent. Therefore, it is important to write the potential in our form $U + W$.

2.3. In the rest of this paper, we consider Λ to be rectágular regions. We consider observables of the form $A_x = \int_{\Delta_x} J(y)\, dy$, where Δ_x is the unit lattice square that contains $x \in R^2$.

Theorem 2.1. Let d be the distance of Δ_x and Δ_y such that $d > l_D$. (a) There exists a constant λ_0 such that for all λ, $0 < \lambda \leqslant \lambda_0$, and all sufficiently small β depending on λ, we have

(i) The infinite-volume limit exists,

$$\lim_{\Lambda \nearrow R^2} \langle A_x \rangle_\Lambda = \langle A_x \rangle, \qquad \lim_{\Lambda \nearrow R^2} \langle A_x A_y \rangle_\Lambda = \langle A_x A_y \rangle$$

(ii) The system screens, i.e., there exists a constant c independent of β such that

$$|\langle A_x A_y \rangle - \langle A_x \rangle \langle A_y \rangle| \leqslant c\tilde{z} \exp(-d/2l_D)$$

(b) For any $l' > l_D$, there exists a constant $\lambda_0(l')$ such that for any $0 < \lambda \leqslant \lambda_0(l')$, the system has screening length l' provided β is smaller than a constant $\beta_0(\lambda)$. Here $\beta_0(\lambda)$ tends to zero as λ tends to zero, and $\lambda_0(l')$ goes to zero as $l' \downarrow l_D$.

Our method of proof also yields the existence of the infinite-volume limit for a product of more than two A's. When $\tilde{z}$ or β is zero, Theorem 2.1 can be proved by explicit computation. From now on, we assume $\tilde{z}$ and β are positive.

3. SINE-GORDON TRANSFORMATION AND THE SHORT-RANGE PART

3.1. Sine-Gordon transformation. Let $d\mu_0(\phi)$ be the Gaussian measure with mean 0 and covariance u. Let

$$e^M = Z_0^{-1} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_n \sum_{e_1 \cdots e_n} \exp\left[-\beta W + \sum_{i=1}^{n} i\beta^{1/2} e_i \phi(x_i)\right] \tag{3.1}$$

We apply the sine-Gordon transformation (see, e.g., Ref. 2) to obtain

$$Z = \frac{1}{Z_0} Z_\Lambda = \int d\mu_0(\phi)\, e^M \tag{3.2}$$

Let $\varepsilon_e(x) = \exp[i\beta^{1/2}\phi(x)e] - 1$. By Mayer's expansion,

$$M(\phi) = \sum_{s=1}^{\infty} \frac{1}{s!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_s$$

$$\sum_{e_1 \cdots e_s} \rho_{e_1,\ldots,e_s}(x_1,\ldots, x_s) \prod_{i=1}^{s} \varepsilon_{e_i}(x_i) \tag{3.3}$$

$$\rho_{e_1,\ldots,e_s}(x_1,\ldots, x_s)$$

$$= \sum_{t \geqslant s} \frac{\tilde{z}^t}{(t-s)!} \int_\Lambda dx_{s+1} \cdots \int_\Lambda dx_t \sum_{e_{s+1} \cdots e_t} (e^{-W})_c \tag{3.4}$$

$(e^{-W})_c$ is the Ursell function of $\exp[-(\beta/2)\sum_{i \neq j} e_i e_j w(x_i, x_j)]$. Theorem 3.1, Theorem 3.2, and (3.7) imply that (3.3) is a convergent series uniformly in Λ if λ is sufficiently small and $\beta < 4\pi$.

For $\mathscr{A}(\phi)$ a functional of ϕ, we define

$$I(\mathscr{A}(\phi)) = \int d\mu_0(\phi)\, e^{M(\phi)} \mathscr{A}(\phi), \qquad \langle \mathscr{A}(\phi) \rangle_\Lambda = Z^{-1} I(\mathscr{A}(\phi)) \tag{3.5}$$

The above ρ's are called truncated correlation functions of the system with pair potentials $e_i w(x_i, x_j) e_j$. In Section 3.2, we shall show that the limit of $\rho_{e_1,\ldots,e_s}(x_1,\ldots, x_s)$ as $\Lambda \nearrow R^2$ exists, for distinct $x_1, x_2,\ldots, x_s$. We also denote the limit by $\rho_{e_1,\ldots,e_s}(x_1,\ldots, x_s)$. Both ρ_{+1} and ρ_{-1} are independent of x when $\Lambda \nearrow R^2$. We write $\tilde{l}_D = (2\rho_{+1}\beta)^{-1/2}$. The second term of (3.20) goes to zero uniformly in β, $0 \leqslant \beta \leqslant 2\pi$, as λ goes to zero. Therefore,

$$\lim \tilde{l}_D = l_D \tag{3.6}$$

uniformly in β, $0 \leqslant \beta \leqslant 2\pi$, as $\lambda \to 0$.

3.2. The short-range part. For our short-range potentials, we shall obtain estimates analogous to the estimates in Appendix 1 of Ref. 2. We use an "iterated Mayer expansion"[3] of the truncated correlation function ρ, and obtain a sufficient condition similar to (A1.6) of Ref. 2 for convergence of the expansion.

Let T^t be the set of all tree graphs on $\{1,\ldots, t\}$. Let $1 \leqslant s \leqslant t$ and $\eta \in T^t$. By removing branches of the tree that contain no vertices of $1,\ldots, s$ and also removing vertices $s+1,\ldots, t$ that join exactly two lines, a unique minimal augmented tree graph η^A of order s is determined. We denote the set of all minimal augmented trees of order s by A^s. We write $\eta \in \eta^A$ if η determines η^A.

Let $a_1,\ldots, a_s$ be lattice squares in R^2. For any $\eta \in T^t$, we define

$$L_\eta = L_\eta(a_1,\ldots, a_s) = \inf \sum_{i,j \in \eta} d(x_i, x_j)$$

where the infimum is taken over the set $x_j \in a_j$, for $j = 1, 2, \ldots, s$, and $x_j \in R^2$ for $j = s+1, \ldots, t$. We note that $L_\eta = L_{\eta A}$. A useful property of $L_{\eta A}$ is

$$\sum_{a_2,\ldots,a_n} \exp[-\alpha L_{\eta A}(a_1,\ldots, a_n)/\tilde{l}_D] \leqslant c_\alpha^{n-1} \tag{3.7}$$

for some c_α where $c_\alpha \to 1$ as $\alpha \to \infty$. Here the a_i are lattice squares of size $\tilde{l}_D$.

For $\gamma > 1$, we write $w = \sum_{K=0}^{\infty} w^{(K)}$, where

$$w^{(K)} = (-\Delta_A + \gamma^{2K}\lambda^{-2} l_D^{-2})^{-1} - (-\Delta_A + \gamma^{2K+2}\lambda^{-2} l_D^{-2})^{-1}$$

$w^{(K)}$ has the stability property: Let $B^{(K)} = \beta \log \gamma/4\pi$; then

$$\sum_{1 \leqslant i \neq j \leqslant n} e_i e_j \beta w^{(K)}(x_i, x_j) \geqslant -nB^{(K)}$$

We define $B^{\leqslant K} = \sum_{l=0}^{K} B^{(l)}$ and

$$\|w^{(K)}\|_\alpha = \int |w^{(K)}(0, x)| \exp(\alpha\, |x|/\tilde{l}_D)\, dx$$

$$\|w\| = \sum_{K=0}^{\infty} \|w^{(K)}\|_\alpha \exp(2B^{\leqslant K})$$

By Theorem 2.5(a) in Ref. 3, when $1 \leqslant s \leqslant t$, and $2 \leqslant t$,

$$(e^{-W})_c(x_1,\ldots, x_t) = \sum_{\eta A \in A^s} Q_{\eta A}(t) \tag{3.8}$$

$$Q_{\eta A}(t) = \sum_{\eta \in \eta A \cap T^t} \sum_K \prod_{ij \in \eta} [-e_i e_j w^{(K_{ij})}(x_i, x_j)\beta] \int dP_{\eta,K}(r)\, e^{-W(r)} \tag{3.9}$$

Here $K = (K_{ij})$, $ij \in \eta$, and $dP_{\eta,K}(r)$ is a probability measure depending on η and K. Here $W(r)$ is an interacting potential depending on r.

By Theorem 2.2 in Ref. 3, if $t \geqslant 2$, $|Q_{\eta A}(t)|$ is bounded by

$$\sum_{\eta \in \eta A \cap T^t} \sum_K \prod_{ij \in \eta} |\beta w^{(K_{ij})}(x_i, x_j)| \exp(2B^{\leqslant K_{ij}}) \tag{3.10}$$

We put $Q_{\eta A}(1) = 1$ and define, for $s \geqslant 1$,

$$\rho^{\eta A}_{e_1,\ldots,e_s}(x_1 \cdots x_s) = \sum_{t \geqslant s}^{\infty} \frac{\tilde{z}^t}{(t-s)!} \int_A dx_{s+1} \cdots \int_A dx_t \sum_{e_{s+1},\ldots,e_t} Q_{\eta A}(t) \tag{3.11}$$

By (3.4) and (3.11),

$$\rho_{e_1,\ldots,e_s}(x_1,\ldots, x_s) = \sum_{\eta^A \in A^s} \rho^{\eta^A}_{e_1,\ldots,e_s}(x_1,\ldots, x_s) \tag{3.12}$$

Let $a_1, \ldots, a_s$ be lattice squares of size $\tilde{l}_D$. We shall estimate the following quantities. For $\eta^A \in A^s$, we define

$$\mathscr{E}_{\eta^A}(a_1, \ldots, a_s) = \frac{1}{s!} \sum_{e_1, \ldots, e_s} \int_{a_1} dx_1 \cdots \int_{a_s} dx_s \, \rho^{\eta^A}_{e_1, \ldots, e_s}(x_1, \ldots, x_s)\, \varepsilon_{e_1}(x_1) \cdots \varepsilon_{e_s}(x_s) \tag{3.13}$$

$$\mathscr{E}(a_1, \ldots, a_s) = \sum_{\eta^A} \mathscr{E}_{\eta^A}(a_1, \ldots, a_s) \tag{3.14}$$

$$k_s(a_i) = \int_{a_1} dx_1 \cdots \int_{a_s} dx_s \, |\rho_{e_1, \ldots, e_s}(x_1, \ldots, x_s)| \tag{3.15}$$

$$k_{\eta^A}(a_i) = \int_{a_1} dx_1 \cdots \int_{a_s} dx_s \, |\rho^{\eta^A}_{e_1, \ldots, e_s}(x_1, \ldots, x_s)| \tag{3.16}$$

The estimates for (3.13)–(3.16) can be easily obtained by the following theorems.

Theorem 3.1. If $\kappa = 4e\tilde{z}\beta\, \|w\| < 1/2$, then there exist constants b_{η^A} and $c_s(\alpha)$ such that

$$b_{\eta^A} \geqslant 0, \qquad \sum_{\eta^A} b_{\eta^A} = 1$$

$$|k_{\eta^A}(a_i)| \leqslant c_s(\alpha)\, b_{\eta^A} \exp[-\alpha L_{\eta^A}(a_i)/\tilde{l}_D]$$

$$c_s(\alpha) \leqslant \tilde{l}_D^2\, 2s!\, \kappa^s / e\beta\, \|w\|$$

Theorem 3.2. For any α, if $0 \leqslant \delta < 4\pi$, then $\kappa \to 0$ uniformly in $\tilde{z}$, β, $0 \leqslant \beta \leqslant \delta$, as $\lambda \to 0$.

Remark. In view of Theorems 3.1 and 3.2, from now on we always choose λ so small that we can set $\alpha = 1$.

Corollary 3.3. $\mathscr{E}_{\eta^A}(a_1, \ldots, a_s)$ is bounded by

$$\tilde{l}_D^2\, 2^{s+1} \kappa^s (\|w\|\, e\beta)^{-1} b_{\eta^A} \exp(-\alpha L_{\eta^A}/\tilde{l}_D) \tag{3.17}$$

Proof of Theorem 3.1. We shall prove the theorem for $\tilde{l}_D = 1$. For general $\tilde{l}_D$, the proof is straightforward. When $x_i \in a_i$ for $i = 1, \ldots, s$ and $x_i \in R^2$ for $i = s+1, \ldots, t$, we have

$$\prod_{ij \in \eta} \exp(\alpha\, |x_i - x_j|) \geqslant \exp(\alpha L_{\eta^A}) \tag{3.18}$$

Applying bounds (3.10) and (3.18) to (3.16), we get

$$|k_{\eta^A}(a_i)| \leqslant \sum_{t \geqslant 2, t \geqslant s} \frac{\tilde{z}^t}{(t-s)!} \int_{a_1} dx_1 \int_A dx_2 \cdots \int_A dx_t$$

$$\times \sum_{e_{s+1},\ldots,e_t} \sum_{\eta \in \eta^A \cap T^t} \sum_K \prod_{ij \in \eta} |\beta w^{(K_{ij})}(x_i, x_j)|$$

$$\times \exp(\alpha\,|x_i - x_j|) \exp(2B^{\leqslant K_{ij}}) \exp(-\alpha L_{\eta^A}) + 2\tilde{z}\,\delta_1(s) \qquad (3.19)$$

Integrating over $dx_1, \ldots, dx_t$ and summing over $e_{s+1}, \ldots, e_t$, we get

$$|k_{\eta^A}(a_i)| \leqslant 2\tilde{z}\,\delta_1(s) + \sum_{t \geqslant 2, t \geqslant s} \frac{\tilde{z}^t \beta^{t-1}}{(t-s)!} \|w\|^{t-1} 2^{t-s} t^{t-2}$$

$$\times \sum_{\eta \in \eta^A \cap T^t} t^{2-t} \exp(-\alpha L_{\eta^A}) \qquad (3.20)$$

We use $t!/(t-s)!\,s! \leqslant 2^t$, for all $1 \leqslant s \leqslant t$, and Stirling's formula; $|k_{\eta^A}(a_i)|$ is bounded by

$$2\tilde{z}\,\delta_1(s) + 2^{-s}\kappa^s s!\,(\|w\|\,\beta e)^{-1}(1-\kappa)^{-1} b_{\eta^A} \exp(-\alpha L_{\eta^A}) \qquad (3.21)$$

where

$$b_{\eta^A} = \sum_{t \geqslant s} \kappa^{t-s}(1-\kappa) \sum_{\eta \in \eta^A \cap T^t} t^{2-t}$$

By Cayley's theorem,

$$\sum_{\eta^A} b_{\eta^A} = \sum_{t \geqslant s} \kappa^{t-s}(1-\kappa) = 1$$

By the assumption $(1-\kappa)^{-1} < 2$, (3.21) implies our theorem for $s \geqslant 2$. For $s = 1$, our theorem also holds because, again by (3.21),

$$|k_{\eta^A}(a_i)| \leqslant 6\tilde{z} \leqslant 2s!\,\kappa^s(\|w\|\,\beta e)^{-1} b_{\eta^A} \exp(-\alpha L_{\eta^A})$$

Proof of Theorem 3.2. Let $\|\bar{w}\|$ be the same as $\|w\|$ but with $\tilde{l}_D$ replaced by l_D. Let $\bar{\kappa} = 4e\tilde{z}\beta\,\|\bar{w}\|$. By the same argument as in Theorem 3.1, if $\bar{\kappa} < 1/2$, then

$$|k_{\eta^A}(a_i)| \leqslant c_s(\alpha)\, b_{\eta^A} \exp[-\alpha L_{\eta^A}(a_i)/l_D] \qquad (3.22)$$

By the definition of $\tilde{l}_D$, (3.22) implies that if $\bar{\kappa} < 1/2$, then $\tilde{l}_D \geqslant c l_D$, where c is a constant independent of $\tilde{z}, \beta, \lambda$. Therefore, to prove Theorem 3.2, it is sufficient to prove an anlogous theorem for $\bar{\kappa}$.

We shall prove the theorem for the case $l_{\mathrm{D}}=1$. The proof for general l_{D} is straightforward.

Let $c=\lambda^{-1}l_{\mathrm{D}}^{-1}$. Since

$$|w^{(K)}(0, y)| \leqslant (\log \gamma)(2\pi)^{-1} \exp(-c\gamma^{K}\,|y|)$$

it follows that $\|w^{(K)}\|_{\alpha}$ is bounded by $(2 \log \gamma)(\alpha - c\gamma^{K})^{-2}$.

For any α, we can choose λ so small that $c>2\alpha$. Then $(c\gamma^{K}-\alpha)^{2} \geqslant (c\gamma^{K}/2)^{2}$. Recall

$$B^{(K)}=(\beta \log \gamma)/4\pi, \qquad B^{\leqslant K}=(4\pi)^{-1}(K+1)\beta \log \gamma$$

Therefore, $\|\bar{w}\|$ may be bounded by

$$c^{-2} 8 \log \gamma \sum_{k=0}^{\infty} \gamma^{-2K} \exp[(2\pi)^{-1}(K+1)\beta \log \gamma]$$

$$= c^{-2} 8(\log \gamma)\, \gamma^{\beta/2\pi} \sum_{K=0}^{\infty} \gamma^{(K\beta/2\pi)-2K}$$

Since $\beta \leqslant \delta$ and $\delta < 4\pi$, the above series converges, and $\|\bar{w}\|$ is bounded by $c^{-2}8(\log \gamma)\, \gamma^{\beta/2\pi}(1-\gamma^{-2+\beta/2\pi})^{-1}$, which goes to zero uniformly in $\beta, 0 \leqslant \beta \leqslant \delta$, as λ goes to 0.

Let $x \in R^{2}$ and Δ_{x} be the unit lattice square that contains x. We define

$$\|w\|' = \sum_{K} \left[\int |w^{(K)}(0, y)|^{2} \exp(2\alpha\,|y|/\bar{l}_{\mathrm{D}}) \right]^{1/2} \exp(2B^{\leqslant K})$$

$$\|w\|'' = \max\{\|w\|, \|w\|'^{2}\}$$

Using the same method as that in Theorem 3.2, we can prove that $\kappa''=4e\tilde{z}\beta\,\|w\|''$ goes to zero, uniformly in $\tilde{z}, \beta$, for $0 \leqslant \beta \leqslant \delta$, $\delta<2\pi$, as λ goes to zero. For the next theorem, we let $d=\operatorname{dist}(\Delta_{x}, \Delta_{y})$.

Theorem 3.4. Suppose $0<\beta<2\pi$ and $d \geqslant l_{\mathrm{D}}$. There exists a constant $c(\beta)$ such that, if λ is sufficiently small depending on α, then

$$\int_{\Delta_x} dx_1 \int_{\Delta_y} dx_2 \int_{R^2} dx_3 \cdots \int_{R^2} dx_s\, |\rho^{\eta^A}_{e_1,\ldots,e_s}(x_1,\ldots,x_s)|$$

$$\leqslant c(\beta)\, c_s'(\alpha)\, b_{\eta^A} \exp[-\alpha L_{\eta^A}(\Delta_x, \Delta_y)/\bar{l}_{\mathrm{D}}] \qquad (3.23)$$

Here $c_s'(\alpha) \leqslant 2s!\,(\kappa'')^{s}/e\beta(\|w\|'')^{2}$ and $c(\beta)<\infty$ as $\beta \to 0$.

Proof. We see that (3.23) differs from (3.16) only in the domain of the integration. By (3.10), (3.23) is bounded by the right side of (3.19), with

$L_{\eta^A}(a_i)$ replaced by $L_{\eta^A}(\Delta_x, \Delta_y)$, and $(a_1, \Lambda, ..., \Lambda)$ replaced by $(\Delta_x, \Delta_y, R^2, ..., R^2)$. Therefore, to prove the theorem, it is sufficient to prove (for $\tilde{l}_D = 1$) that there exists $c(\beta)$ such that

$$\sum_K \sum_{e_{s+1},...,e_t} \int_{\Delta_x} dx_1 \int_{\Delta_y} dx_2 \cdots \int_{R^2} dx_t \prod_{ij \in \eta} |\beta w^{(K_{ij})}(x_i, x_j)| \times \exp(\alpha |x_i - x_j|) \exp(2B^{\leqslant K_{ij}}) \tag{3.24}$$

is bounded by

$$c(\beta)(\|w\|'')^{t-2}\beta^{t-1}2^{t-s} \tag{3.25}$$

for all $\eta \in \eta^A \cap T^t$. To prove this, we consider the following two cases.

Case 1. If there is a bond between 1 and 2 in η, then (3.24) is bounded by

$$\beta^{t-1} \|w\|^{t-2} c'(w) \tag{3.26}$$

with

$$c'(w) = (2\pi)^{-1}(\log \gamma) \exp[\beta(2\pi)^{-1} \log \gamma] \times \sum_K \exp(K\beta \log \gamma/2\pi) \exp[-(d/l_D)(\gamma^K/\lambda - \alpha)]$$

If we choose $\lambda \leqslant \min\{\gamma/2\alpha, 1/4\}$ and $d \geqslant l_D$, then

$$c'(w) \leqslant c(\beta) = (2\pi)^{-1}(\log \gamma) \exp[(2\pi)^{-1} \log \gamma] \times \sum_K \exp(K\beta \log \gamma/2\pi) \exp(-\gamma^K/2) \tag{3.27}$$

$c(\beta) < \infty$ for all $0 \leqslant \beta < 2\pi$ and $\lim c(\beta) = c(0)$, as β goes to zero.

Case 2. If there is no bond between 1 and 2 in η, we use the Schwarz inequality to get an upper bound for (3.24) by

$$\beta^{t-1}(\|w\|')^2 \|w\|^{t-3} \tag{3.28}$$

In both cases, (3.24) is bounded by (3.25).

Using the same argument as that in Theorem 3.4, we also obtain the following estimates. Let

$$f(d) = \sum_{k=0}^{\infty} \exp(k\beta \log \gamma/2\pi) \exp(-d\gamma^K/2l_D) \tag{3.29}$$

Suppose $\lambda \leqslant \min\{\gamma/2\alpha, 1/4\}$; then

$$\int_{R^2} dx_3 \cdots \int_{R^2} dx_s \, |\rho^{\eta^A}_{e_1 \cdots e_s}(x_1, ..., x_s)|$$
$$\leqslant f(|x_1 - x_2|) \, c'_s(\alpha) \, b_{\eta^A} \exp(-\alpha \, |x_1 - x_2|/\bar{l}_D) \tag{3.30}$$

We note that $f(0) = \infty$ and $f(d) < \infty$ if $0 \leqslant \beta < 2\pi$ and $d > 0$. A useful fact is that

$$\int_0^1 f(r) \, dr < \infty \qquad \text{if} \quad 0 \leqslant \beta < 2\pi \tag{3.31}$$

4. THE LONG-RANGE PART

We shall prove Theorem 2.1 by applying the cluster expansion and Peierl's expansion to the long-range part of the potentials. The proofs are almost the same as the proofs in Ref. 2 except that we replace (1) the three-dimensional objects by analogous two-dimensional objects and (2) the estimates in Appendix 1 of Ref. 2 by the estimates in Section 3.2. We set $\bar{l}_D = 1$ in this section. The expansion for general $\bar{l}_D$ is straightforward.

4.1. Peierl's expansion. Let Λ be a rectangular domain that is a union of closed unit squares. In this paper, lattice squares mean closed lattice squares. Let $L \ll 1 \ll L'$. Let $\{\Omega_\alpha\}$ be lattice squares in Λ with size L, and $\{\Delta_\alpha\}$ be unit lattice squares in Λ. Let $\tau = 2\pi\beta^{-1/2}$. Let h be a function on R^2 with values integral multiples of τ, such that h is constant on the interior of each Ω_α and zero on Λ^c. The Peierl's contour $\sum(h)$ of h is the set of all discontinuities of h. Let $\sum^\Lambda(h)$ be the set of unit lattice squares in Λ whose distance from $\sum(h)$ is less than L'. We set

$$A_\alpha = L^{-2} \int_{\Omega_\alpha} \phi(x) \, dx$$
$$\delta(x) = \phi(x) - A_\alpha(x) \qquad \text{for} \quad x \in \Omega_\alpha$$

Let $g = g_h$ be a function on R^2 depending on h. We perform a translation $\phi = \psi + g$. We define the following covariance C_0 and C:

$$C_0^{-1} = u^{-1} + \bar{l}_D^{-2} \tag{4.1}$$
$$C^{-1} = C_0^{-1} + v \tag{4.2}$$
$$\mathscr{L}_c = \bar{l}_D^{-2} C_0 \tag{4.3}$$

where $v/2 = \sum_{e_1, e_2} \beta e_1 e_2 \rho_{e_1, e_2}(x_1, x_2)/2$ is the quadratic part in ψ of

$M(\psi + g)$. Let $d\mu$ be the Gaussian measure with mean 0 and covariance C. Then

$$Z = \sum_h N \int d\mu(\psi)\, e^E e^G e^R \tag{4.4}$$

$$N = \int d\mu_0(\psi) \exp\left(-2^{-1}\bar{l}_D^{-2} \int \psi^2 - 2^{-1} \int \psi v \psi\right) \tag{4.5}$$

$$E = M(\phi) - \sum_e \rho_e \int \varepsilon_e(x)\, dx + 2^{-1} \int \psi v \psi \tag{4.6}$$

Let $\omega_e(\phi) = \exp(i\beta^{1/2} e\phi) - 1$. Then

$$e^G = \exp\left\{\sum_e \rho_e \int [\omega_e(\delta) - i\beta^{1/2} e\delta + \beta\delta^2 2^{-1}]\right\}$$
$$\times \exp\left\{\sum_e \rho_e \int \omega_e(A)[\omega_e(\delta) - i\beta^{1/2} e\delta]\right\} \prod_\alpha r_\alpha(A_\alpha) \tag{4.7}$$

$$r(A) = \exp\left[\sum_e \rho_e \omega_e(A)\, L^2\right] \left\{\sum_n \exp[-L^2(A - n\tau)^2/2\bar{l}_D^2]\right\}^{-1} \tag{4.8}$$

$$R = -F_1 - F_2 \tag{4.9}$$

$$F_1 = 2^{-1}\bar{l}_D^{-2} \int (g-h)^2 + 2^{-1} \int g u^{-1} g \tag{4.10}$$

$$F_2 = \int \psi c_0^{-1}(g - g_c) \tag{4.11}$$

The above integrations on R^2 are over Λ.

By (7.19) of Ref. 2, which works also for the two-dimensional case, it is possible to define g such that: (1) $g = h$ outside $\sum^\Lambda$. (2) Inside any connected component of $\sum^\Lambda$, g depends only on h inside the same component. (3) g is in the domain of C_0^{-1}. (4) $\int \psi C_0^{-1}(g - g_c)$ can be estimated to be small, in the sense of Proposition 5.4.

4.2. The cluster expansion. We shall use the same expansion formula as that in Ref. 2. We use the following notations.

For a fixed h, let $\tilde{Y} = \tilde{Y}(h)$ be the set whose elements are either connected components of $\sum^\Lambda(h)$ or closed unit lattice squares in $\Lambda \backslash$interior of $\sum^\Lambda(h)$. Let $\bar{y}$ be a sequence of sets $Y_1, Y_2, \ldots, Y_n$, where Y_i is union of elements of $\tilde{Y}$ and $Y_i \cap Y_j = \varnothing$, for all $i \neq j$. We let $X_1 = Y_1$, $X_i = Y_i \cup X_{i-1}$, and $X_n = X$. For any $Y \subset \Lambda$, we write $Y^c = \Lambda \backslash Y$. By $d\mu_s(\psi)$

we mean a Gaussian measure with mean 0 and covariance $C(x, y, s) = p(x, y, s)\, C(x, y)$, where

$$p(x, y, s) = \sum_{1 \leqslant i < j \leqslant n+1} s_i s_{i+1} \cdots s_{j-1} 1_i(x)\, 1_j(y) + \sum_{1 \leqslant j < i \leqslant n+1} s_j s_{j+1} \cdots s_{i-1} 1_i(x)\, 1_j(y) + \sum_{1 \leqslant i \leqslant n+1} 1_i(x)\, 1_i(y) \tag{4.12}$$

Here $1_i(x)$ is the characteristic function of Y_i, $1_{n+1}(x)$ is the characteristic function of $(\bigcup_{i=1}^{n} Y_i)^c$, and $s_n = 0$.

We put, for a sequence of unit lattice squares $a_1, a_2, \ldots, a_t$,

$$\mathscr{E}(a_1, \ldots, a_t) = \frac{1}{t!} \sum_{e_1, \ldots, e_t} \int_{a_1} dx_1 \cdots \int_{a_t} dx_t \times \rho_{e_1, \ldots, e_t}(x_1, \ldots, x_t)\, \varepsilon_{e_1}(x_1) \cdots \varepsilon_{e_t}(x_t) \tag{4.13}$$

By the definition of $E(\Lambda)$, $E(\Lambda)$ may be written as a sum of terms of $\mathscr{E}(a_1, \ldots, a_t)$ and a term where $\varepsilon_e(x)$ is replaced by $\psi e \beta^{1/2}$ when $t = 2$. We define $E(Y)$ to be the same sum of $\mathscr{E}(a_1, \ldots, a_t)$ as in $E(\Lambda)$, but a_i must be in Y, for all i. The term $E(X, s)$ is the same sum of terms as added to $E(X)$, but each term $\mathscr{E}(a_1, \ldots, a_n)$ is multiplied by $\prod_{i \in I} s_i$, where $i \in I$ if $1 \leqslant i \leqslant n-1$ and if there exist α, β such that $1 \leqslant \alpha, \beta \leqslant t$, $a_\alpha \subset Y_{i+1}$, $a_\beta \subset X_i$.

As in Ref. 2, we define the following operators:

$$\kappa(\bar{y}, s) = \prod_{i=1}^{n-1} \kappa(i)$$

$$\kappa(i) = \frac{d}{ds_i} E^{(i)}(X, s) + \int_{Y_{i+1}} dx \int_{X_i} dy \left(\frac{d}{ds_i} C(x, y, s) \right) \times \left[\left(\frac{\delta}{\delta\psi(x)} + \frac{\delta E(X, s)}{\delta\psi(x)} \right) \left(\frac{\delta}{\delta\psi(y)} + \frac{\delta(E(X, s))}{\delta\psi(y)} \right) \right]^{(i)}$$

We write $Y' \prec Y$ if $Y' \subset Y$ and Y is the smallest union of sets in $\tilde{Y}$ that contains Y'. The $E^{(i)}(X, s)$ contains $\mathscr{E}(a_1, \ldots, a_t)$ in $E(X, s)$ with the same multiplication of s, and if $\bigcup_{k=1}^{t} a_k \backslash X_i \prec Y_{i+1}$. The (i) on the bracket means that when we expand the bracket into four terms, in each term, if there are sequences $a_1, \ldots, a_t$ and $a'_1, \ldots, a'_r$ contributing to $E(X, s)$'s, the sequences must satisfy $(\bigcup_{k=1}^{t} a_k) \cup (\bigcup_{k=1}^{r} a'_k) \backslash X_i \prec Y_{i+1}$.

Let $\mathscr{A}$ be a functional of ϕ, periodic in τ. By Section 8 of Ref. 2 the expansion formulas are as follows:

$$\frac{1}{Z_0} I(\mathscr{A}(\phi)) = \sum_X \mathscr{K}(X)\, Z'(\Lambda, X) \tag{4.14}$$

$$\mathscr{K}(X) = \sum_{\bar{y}} \sum_h \int ds \int d\mu_s(\psi)\, e^{E(X,s)} \kappa(\bar{y}, s)\, e^{G(X)}\, e^{R(X)} \mathscr{A} \tag{4.15}$$

$$Z'(\Lambda, X) = \sum_h N \int d\mu(\psi)\, e^{E(X^c)}\, e^{G(X^c)}\, e^{R(X^c)} \tag{4.16}$$

In (4.14), X runs over all unions of lattice squares. In (4.15), $\bar{y}$ is a sequence of sets $Y_1,..., Y_n$, where the Y_i are disjoint and $\bigcup_{i=1}^n Y_i = X$. For a fixed $\bar{y}$, we sum over all those h such that $\bar{y}$ is compatible with h and Y_1 is the smalles union of sets in $\tilde{Y}(h)$ that contains the support of $\mathscr{A}$.

5. CONVERGENCE OF THE EXPANSION

5.1. We shall prove that our expansions of Section 4 converge in the following sense. Let Δ_i, $i = 1,..., w_1$ be unit lattice squares and a_i, $i = w_1 + 1,..., w_1 + w_2$ be lattice squares of size $\bar{l}_D$. Let X_1 be the minimal union of lattice squares of size $\bar{l}_D$ that contains $\bigcup \Delta_i \cup a_i$ and $X_0 = \bigcup a_i$. The notation $|X|$ means the number of lattice squares of size $\bar{l}_D$ in X. We consider $\mathscr{A}$ of the following form:

$$\mathscr{A} = \prod_i \int_{\Delta_i} \exp[i\beta^{1/2}\phi(x_i) e_i] \prod_j \int_{a_j} \varepsilon_{e_j}(x_j)\, \zeta(x_1,..., x_{w_1+w_2}) \tag{5.1}$$

We shall fix δ_1 such that $0 < \delta_1 < 1/2$.

Theorem 5.1. If λ, L are sufficiently small and L' is sufficiently large according to δ_1, then for any $c'_A > 0$, there exist c_3, c_A (independent of β) such that

$$\begin{aligned} \sum |\mathscr{K}(X)| \exp(c'_A |X|) &\\ \leqslant c_3^{w_1+w_2} \|\zeta\|\, \beta^{|X_0|/2} &\exp(c_A |X_1|) \\ \times &\exp[-(1-2\delta_1) \operatorname{dist}(X_1, W)/\bar{l}_D] \end{aligned} \tag{5.2}$$

for β sufficiently small according to c'_A, λ, L', L, and δ_1. Here

$$\|\zeta\| = \prod \int_{a_i} dx_i \prod \int_{\Delta_j} dx_j\, |\zeta(x_1,..., x_{w_1+w_2})|$$

the summation is over all X such that $X \supset X_1$ and $X \cap W \neq \phi$. Moreover, if the above sum is restricted to $X \supset X_1$ and $X \neq X_1$, then $c_3^{w_1+w_2}$ can be replaced by $cc_3^{w_1+w_2}$, where $c \to 0$ as $\beta \to 0$.

Theorem 5.1 is the two-dimensional version of Lemma 9.4 of Ref. 2. Since our $\mathscr{A}$ differs from the $\mathscr{A}$ in Ref. 2, a factor $\beta^{|X_0|/2}$ is included in our estimate.

We shall prove Theorem 5.1 for $\tilde{l}_D = 1$. For general $\tilde{l}_D$, we can use the change of variables $\tilde{l}_D \to \tilde{l}_D/l$, $\tilde{z} \to \tilde{z}l^2$, $\beta \to \beta$, $\lambda \to \lambda$, $x \to x/l$, $\phi \to \phi$, $\delta_1 \to \delta_1$, and so on.

5.2. According to our expansion formula (4.15), the left-hand side of (5.2) can be written as sums over:

1. n: the length of sequence $\bar{y}$.
2. (m_i), $i = 1,\ldots, n$: Y_i is a union of m_i sets Y_{ij}.
3. (Y_{ij}): choices of sets from $\tilde{Y}$.
4. h: h should be compatible with Y_{ij}.
5. $\int ds$: integration over ds.
6. T: the label increasing tree graphs. We write

$$\prod_i \int_{Y_{i+1}} \int_{X_i} = \sum_T \prod_i \int_{Y_{i+1}} \int_{Y_{T(i+1)}}$$

T is a mapping from $\{1,\ldots, n\}$ to $\{1,\ldots, n\}$ such that $T(i) < i$.

7. Types of terms: $\kappa(i)$ is a sum of five types of terms.
8. (t): E's are sums over $t \geqslant 2$ of terms as in (4.13).
9. Δ_i', Δ_i'': $i = 1,\ldots, n-1$. We write

$$\int_{Y_{i+1}} dx \int_{Y_{T(i+1)}} dy = \sum_{\Delta_i' \Delta_i''} \int_{\Delta_i'} dx \int_{\Delta_i''} dy$$

10. $(a) = a_1,\ldots, a_t$ in (4.13) is a sequence of unit lattice squares. (a) must be compatible with Y_{ij}, Δ_i', Δ_i''.

By (3.14), $\mathscr{E}(a_i) = \sum_{\eta^A} \mathscr{E}_{\eta^A}(a_i)$. We define a formal operator $e^{\delta_2 L_0}$ acting on $\mathscr{E}(a_i)$ or their derivatives $[\prod_j \int_{\Delta_j} \delta/\delta\psi(x_j)]\, \mathscr{E}(a_i)$ as follows. Whenever $e^{\delta_2 L_0}$ meets $\mathscr{E}(a_i)$ that are taken from the right side of $e^{\delta_2 L_0}$, then

$$e^{\delta_2 L_0}\colon \quad \mathscr{E}(a_i) \to \sum_{\eta^A} \exp(\delta_2 L_{\eta^A})\, \mathscr{E}_{\eta^A}(a_i).$$

The operator $e^{\gamma O_0}$ is defined to be a multiplication by $e^{\gamma t}$ if $\mathscr{E}(a_i) = \mathscr{E}(a_1,\ldots, a_t)$.

We define operators $\kappa'' = \prod_{i=1}^{n+1} \kappa''(i)$ as follows. $\kappa''(i)$ is one of the following five operators depending on the type in summation 7. The five operators are

$$e^{rO_0} e^{\delta_2 L_0} \mathscr{E}(a_1,..., a_t)$$

$$\int_{\Delta_i''} dx \int_{\Delta_i'} dy \frac{\delta}{\delta\psi(x)} C(x, y) \frac{\delta}{\delta\psi(y)}$$

$$e^{rO_0} e^{\delta_2 L_0} \left(\int_{\Delta_i''} dx \int_{\Delta_i'} dy \frac{\delta \mathscr{E}_1}{\delta\psi(x)} C(x, y) \frac{\delta \mathscr{E}_2}{\delta\psi(y)} \right)$$

$$e^{rO_0} e^{\delta_2 L_0} \left(\int_{\Delta_i''} dx \int_{\Delta_i'} dy \frac{\delta \mathscr{E}}{\delta\psi(x)} C(x, y) \frac{\delta}{\delta\psi(y)} \right)$$

$$e^{rO_0} e^{\delta_2 L_0} \left(\int_{\Delta_i''} dx \int_{\Delta_i'} dy \frac{\delta}{\delta\psi(x)} C(x, y) \frac{\delta \mathscr{E}}{\delta\psi(y)} \right)$$

Here, for each sequence $a_1,..., a_t$ in $\mathscr{E}$, we have the restrictions $\bigcup a_j \subset X_{i+1}$, $\Delta_i' \cup \Delta_i'' \subset \bigcup a_j$ and $\bigcup a_j \cap Y_{i+1,s} \neq \varnothing$, for all $s = 1, 2,..., m_{i+1}$.

Lemma 5.2. Let $1/2 > \delta_1 > 0$ be fixed. For any $\delta_2, c'_A, c_A > 0$, there exist c'_B, r (in the definition of κ'') such that if β is sufficiently small, then

$$\begin{aligned} &\sum |\mathscr{K}(X)| \exp(c'_A |X|) \\ &\quad \leqslant \sup \exp(c_A F_1 + c'_B |X|) \\ &\quad\quad \times \exp[(1 - 2\delta_1 + \delta_2) d] \exp[-(1 - 2\delta_1) \operatorname{dist}(X_1, W)] \\ &\quad\quad \times \int d\mu_s \, |e^{E(X,s)} \kappa'' e^{G(X)} e^{R(X)} \mathscr{A}|_0 \end{aligned} \tag{5.3}$$

where δ_2 is replaced by $1 - 2\delta_1 + \delta_2$ in κ''. The summation in (5.3) is over all X such that $X \supseteq X_1$ and $X \cap W \neq \varnothing$.

We use the subscript 0 on the absolute value sign to mean that absolute value is taken inside the sum that results when all differentiations in κ'' are performed and inside spatial integrals and sums over species. The sup means supremum over all compatible parameters listed in summations 1–10.

Proof of Lemma 5.2. This is identical to Lemma 9.4 in Ref. 2, except that their dimensions are different. We take the proof from there, which successively uses an inequality of the form

$$\left| \int dv(x) \, f(x) \right| \leqslant \left[\int dv(x) \frac{1}{|a(x)|} \right] \sup_x |a(x) \, f(x)| \tag{5.4}$$

where $dv(x)$ is a measure that is one of the ten summations in our list. Except for the summation 4, it is easy to see that the estimates of the rest of the summations remain true for the two-dimensional case. The estimate of summation 4 depends on the following inequality (Lemma 5.2 of Ref. 1): there exist $c>0$ such that

$$F_1(Y, h) \geqslant c \sum_f |\delta h(f)|^2 \tag{5.5}$$

where f runs over all internal lines of the lattice squares of size L in Y and $\delta h(f)$ is the jump of the value of h between the two squares of size L joined at f. Following the proof of Lemma 5.2 of Ref. 1 with R^3 replaced by R^2, we can prove (5.5) by choosing $c=\min\{1/192, L^2/24\}$, while $c=\min\{L/432, L^3/36\}$ for the three-dimensional case.

After we have done the estimates for summations from 1 to 10, we also use the following inequality to include the factor $\exp[-(1-2\delta_1)\operatorname{dist}(X_1, W)]$ for the right side of (5.3):

$$(1-2\delta_1)\operatorname{dist}(X_1, W) \leqslant (1-2\delta_1)(d+L_0+2^{1/2}|X|) \tag{5.6}$$

Here, X is a union of disjoint Y_i, and each Y_i is a union of disjoint Y_{ij} from $\tilde{Y}$. (5.6) can be understood as follows. L_0 controls the distances between Y_i and $2^{1/2}|X| = 2^{1/2}\sum_{ij}|Y_{ij}|$ controls the total distances inside all of the Y_{ij}.

5.3. ***Proof of Theorem 5.1.*** We shall use the same formula as (9.25) of Ref. 2 to estimate the right side of (5.3) and obtain Theorem 5.1. Let

$$\begin{gathered} \tilde{g}\tilde{\kappa}\mathscr{A} = e^{-R} e^{G_2} \kappa'' e^{G_1} e^{G_2} e^{R} \mathscr{A} \\ \delta(x) = (\psi+g-h)(x) - L^{-2}\int_{\Omega_\alpha} (\psi+g-h)(x)\,dx \qquad \text{for} \quad x\in\Omega_\alpha \end{gathered} \tag{5.7}$$

By Hölder's inequality, the right side of (5.3) is bounded by ($\int$ is understood to mean integrations over X)

$$\begin{aligned} &\sup \exp[-(1-2\delta_1)\operatorname{dist}(X_1, W)/\bar{l}_{\mathrm{D}}] \exp[-(1-c_A)F_1] \\ &\quad\times \|\exp(-F_2)\|_{p_2} \left\| \exp\left[E+G_2-2\bar{l}_{\mathrm{D}}^{-2}\int \delta^2\right]\right\|_{p_4} \\ &\quad\times \left\| |\tilde{g}\tilde{\kappa}\mathscr{A}|_0 \exp\left[c'_B|X| + (1-2\delta_1+\delta_2)\,d + 2\bar{l}_{\mathrm{D}}^{-2}\int \delta^2\right]\right\|_p \end{aligned} \tag{5.8}$$

with $p_4^{-1}+p_2^{-1}+p^{-1}=1$.

We shall use the following estimates.

Proposition 5.3. Given $c_2 > 0$ and $p_4 \geqslant 1$, if λ is sufficiently small, then there exists c such that

$$\left\| \exp\left(E + G_2 - 2\tilde{l}_{\mathrm{D}}^{-2} \int \delta^2 \right) \right\|_p \leqslant \exp(c\,|X| + c_2 F_1)$$

Here c goes to ∞ as λ goes to zero.

Proposition 5.4. There exists $c(L')$ such that

$$\|\exp(-F_2)\|_{p_2} \leqslant \exp[p_2 c(L')\, F_1/2]$$

and $c(L')$ becomes arbitrarily small as L' is increased.

Proposition 5.5. Let

$$\gamma < \tilde{l}_{\mathrm{D}}^{-2}, \qquad B = 2^{-1} \int (\psi + g - h)^2 + 2\gamma^{-1} \int \delta^2$$

Given $p_3 \gamma < \tilde{l}_{\mathrm{D}}^{-2}$, if λ and L are sufficiently small and L' is sufficiently large, then there exist c_1, c_2 such that

$$\|\exp \gamma B\|_{p_3} \leqslant \exp(c_1\,|X| + c_2\,|F_1|)$$

Here $c_2 < 1$, and c_1 goes to ∞ as λ goes to zero.

Proposition 5.6. It is possible to choose $\gamma < \tilde{l}_{\mathrm{D}}^{-2}$ such that, for any $p_3 > p$, c'_B, δ_1, δ_2 if β is sufficiently small, then

$$\left\| \,|\tilde{g}\tilde{\kappa}\mathscr{A}|_0 \exp\left[2\tilde{l}_{\mathrm{D}}^{-2} \int \delta^2 + c'_B\,|X| + (1 - 2\delta_1 + \delta_2) d \right] \right\|_p \leqslant Q(\beta, p_3)\, \|\exp \gamma B\|_{p_3} \tag{5.9}$$

Here $Q(\beta, p_3)$ can be estimated as follows.

(i) When $X \neq X_1$, for any $c_1, c_2 > 0$, if β is sufficiently small according to c_1, c_2, then there exist $c(\beta)$, c_3 such that $\lim c(\beta) = 0$ as β goes to zero, and

$$\exp(-c_1 F_1 + c_2\,|X|)\, Q(\beta, p_3) \leqslant c(\beta)\, c_3^{w_1 + w_2}\, \|\zeta\| \tag{5.10}$$

(ii) When $X = X_1$, for any $c_1 > 0$, if β is sufficiently small according to c_1, then there exist c_3, c''_B such that

$$\exp(-c_1 F_1)\, Q(\beta, p_3) \leqslant c_3^{w_1 + w_2} \exp(c''_B\,|X|)\, \beta^{|X_0|/2}\, \|\zeta\| \tag{5.11}$$

Here c_3 is independent of λ and $\lim c''_B = \infty$ as λ goes to zero.

Proof of Theorem 5.1 Assuming Propositions 5.3–5.6. In (5.8), we choose $p-1$ so small that there exists $p_3 > p$ and $p_3 \gamma < \bar{l}_D^{-2}$. By Propositions 5.3–5.6, we get an upper bound for (5.8),

$$\exp[-c_1 F_1 + c_2 |X| - (1-2\delta_1) \operatorname{dist}(X_1, W)/\bar{l}_D] \, Q(\beta, p_3) \quad (5.12)$$

By Proposition 5.6, when $X \neq X_1$, (5.12) is bounded by

$$\|\zeta\| \, c(\beta) \, c_3^{w_1+w_2} \exp[-(1-2\delta_1) \operatorname{dist}(X_1, W)/\bar{l}_D]$$

which goes to zero as β goes to zero. Therefore, when β is sufficiently small, (5.12) is bounded by the bound for the case $X = X_1$; namely,

$$\|\zeta\| \, \beta^{|X_0|/2} c_3^{w_1+w_2} \exp[c_A |X_1| - (1-2\delta_1) \operatorname{dist}(X_1, W)/\bar{l}_D]$$

for some $c_A > 0$. This is the right side of (5.2).

Proof of Proposition 5.3. This is the two-dimensional analogue of Lemma 9.9 of Ref. 2. The arguments in the proof of Lemma 9.9 of Ref. 2 work also for our case: They are based on (1) estimates of $\mathscr{E}(a_i)$, where we have obtained the same type of estimates in Section 3.2, and (2) boundedness from below of the operators C_s^{-1} uniformly in s and λ. This is also true in our case.

Proof of Proposition 5.4. This is the two-dimensional analogue of Lemma 9.5 of Ref. 2. Using exactly the same argument as in Ref. 2, we obtain

$$c(L') = c_1(L) \, c_2 c_3(L')$$

where $c_1(L)$ is the constant in (5.5), which can be chosen arbitrarily small by decreasing L. The term $c_3(L')$ goes to zero exponentially [$\exp(-L'/8)$; see (9.418) of Ref. 2]. The c_2 is $\sup_x \int dy \, |C(x, y)|$, which is bounded by $c_\lambda (1-\delta_1)^{-2}$. Here we have used an estimate: For any $\delta_1 > 0$, we can choose λ so small that

$$|C(x, y)| \leqslant c_\lambda \exp[-(1-\delta_1) |x-y|] \quad (5.13)$$

where c_λ goes to ∞ as λ goes to zero. Therefore, we let λ be small but nonzero, then we take L' large and L small according to c_λ.

Proof of Proposition 5.5. This is the two-dimensional version of Lemma 9.8 of Ref. 2. The main arguments in Ref. 2 also work for our case, except that, instead of assuming that $\|v\| = \sup_x \int |v(x, y)| \, dy$ is small [see (9.622) of Ref. 2], we can prove that $\lim \|v\| = 0$, as λ goes to zero, by the reults in Section 3.2. After this step, the proof proceeds as in Ref. 2.

5.4. *Proof of Proposition 5.6.* This is the two-dimensional version of Section 9.8 of Ref. 2. We shall follow the argument there. We note that our $\mathscr{A}$ is slightly different from the $\mathscr{A}$ in Ref. 2, and we need a factor $(\beta)^{|X_0|/2}$ in our estimate.

Let $H = \exp(2\tilde{l}_{\mathrm{D}}^{-2} \int \delta^2)$; we shall estimate the L^p-norm of

$$H \,|(\kappa'' e^{G_1} e^{G_2} e^{R} \mathscr{A})\, e^{-R} e^{-G_2} e^{-c_1 d} e^{-c_2|X|}|_0 \tag{5.14}$$

Step 1. We count the number of terms resulting from differentiations in κ''. Each derivative $\partial/\partial\psi$ in κ'' can act on one of ε [or $\bar{\varepsilon} = \varepsilon_{e_i}(x_i) - i\beta^{1/2} e_i \psi$, if $t = 2$], e^R, e^{G_1}, e^{G_2}, $\exp[i\beta^{1/2}\phi(x_i)e_i]$ in $\mathscr{A}$. We write $\delta/\delta\psi = \sum_l (\delta/\delta\psi)_l$ where $(\delta/\delta\psi)_l$ can only act on one of the above five types of factors. We write $\kappa'' = \sum_l \kappa_l''$. Let n_α be the number of derivatives localized in Δ_α, w_α be the number of factors in $\mathscr{A}$ that are localized in Δ_α, and m_α be the number of factors of ε or $\bar{\varepsilon}$ that are from the $\mathscr{E}$'s and localized in Δ_α. Then the number of terms resulting from the differentiations is bounded by

$$\prod_\alpha (m_\alpha + w_\alpha + 3)^{n_\alpha} \tag{5.15}$$

We use the "exponential pinning lemma," Lemma 9.10 of Ref. 2, which holds for the two-dimensional case with a change of constants: Given $c' > 0$, there exists c such that (5.15) is bounded by

$$\exp(c' O_0 + c' d)\, c^{\Sigma(n_\alpha + w_\alpha)} \tag{5.16}$$

By (5.15) and (5.16), (5.14) is bounded by

$$\sup_l \|H c^{\Sigma w_\alpha} \exp(c_1 d + c_2' |X|) \times |\kappa_l''[\exp(G_1)\exp(G_2)\exp(R)]\mathscr{A}|_0 \exp(-R)\exp(-G_2)\|_p \tag{5.17}$$

where the constants in κ_l'' have been increased and $c_2' > c_2$.

Step 2. The operator $\exp(c_1 d)\, \kappa_l''$ is of the form

$$\int J(x) \prod_j \frac{\delta}{\delta\psi(x_j)} \prod_i \varepsilon(x_i) \tag{5.18}$$

where $J(x)$ is a product of C's and ρ^{η^A}'s from (3.13) and includes the factors prescribed by $\exp(c_1 d)$, $\exp(\delta L_0)$, and $\exp(rO_0)$. If κ_l'' involves factors $\bar{\varepsilon}$, then some of the ε's should be replaced by $\bar{\varepsilon}$'s. The $\int$ in (5.18) is a combination of multiple integrals, where each one is over a unit lattice square, summation over species, and sum over η^A's.

We substitute (5.18) into (5.17); then (5.17) may be bounded by the following form:

$$\sup_{l} \left\| H \prod_{\alpha} (n_\alpha !)\, c^{w_\alpha} \exp(c_2' \,|X|) \int |\bar{J}(x)| \prod_{\alpha, i} |T_{\alpha,i}| \right\|_p \tag{5.19}$$

where, for each Δ_α, $T_{\alpha,i}$ is equal to 1 or equal to a derivative of one of the following types, labeled by i. The $\bar{J}$ is the J multiplying ζ. Since our $\mathscr{A}$ is different from the $\mathscr{A}$ in Ref. 2, we have included ε from $\mathscr{A}$ in type (iv).

(i) $\exp[G_1(\Delta_\alpha)] = r(A)$.

(ii) $\exp[i\beta^{1/2}\phi(x_i)e_i]$ from $\mathscr{A}$.

(iii) F_2'.

(iv) $\varepsilon_e(x)$, $\psi(x)$, $\bar{\varepsilon}_e(x)$ from $\mathscr{E}$, and $\varepsilon_e(x)$ from $\mathscr{A}$.

(v) $\exp(i\beta^{1/2}e_i A) - 1$.

(vi) $\exp(i\beta^{1/2}e_i\,\delta) - 1 - i\beta^{1/2}\,e_i\delta$ or

$$\exp(i\beta^{1/2}e_i\delta) - 1 - i\beta^{1/2}e_i\delta + \beta e_i^2\delta^2/2$$

Step 3. Bounds on $|T_{\alpha,i}|$. We shall use the scheme in Ref. 2 to bound $|T_{\alpha,i}|$. To bound the nth derivative of type (i), we may choose c_1, c_2, c_3, and $\gamma < \tilde{l}_{\mathrm{D}}^{-2}$ such that

$$|(d^n/dA^n)\, r(A)| \leqslant c_1(c_2\beta^{1/6})^n \exp(c_3 n \log n + L^2\gamma A/2) \tag{5.20}$$

The proof of (5.20) is exactly the same as the proof of Lemma 9.7 in Ref. 2, except that we replace L^3 by L^2. The nth derivative of a term from (ii) is bounded by $(c\beta^{1/2})^n$. The nth derivative of a term from (v) is bounded by $(c\beta^{1/2})^n \cdot 2$ and the nth derivative of a term from (vi) is bounded by $c\beta(|\delta|^{2-n} + |g-h|^{2-n})$ if $n<2$, or $(c\beta^{1/2})^n$ if $n \geqslant 2$. The ψ in (iv) is bounded by $|\psi|$. To bound ε_e and $\bar{\varepsilon}_e$ in (iv), we divide the lattice squares into two classes. Class A is the set of lattice squares in which $g = h$. Class B consists of the remaining lattice squares. In class A squares, $\bar{\varepsilon}_e$ is bounded by $c\beta\,|\psi|^{2-n}$ if $n<2$, $(c\beta^{1/2})^n$ if $n \geqslant 2$. In class B squares, $\bar{\varepsilon}_e$ is bounded by $2 + c\beta^{1/2}|\psi|$ if $n=0$, and $(c\beta^{1/2})^n$ if $n \geqslant 1$. The nth derivative of ε_e is bounded by $(c\beta^{1/2})^n$ if $n \geqslant 1$. For undifferentiated ε's, we separate them into distinguished ones and undistinguished ones. We bound an undistinguished ε by 2 and a distinguished ε by $c\beta^{1/2}|\psi|$. We first choose a distinguished ε that is localized in each lattice square in class A, then choose three more distinguished ε's localized in class A from each $\mathscr{E}$. If this choice is impossible, then we choose as many as we can.

The L^p-norm of the product of ψ's and δ's resulting from bounds of type (iv) and (vi) can be estimated by Wick's theorem: Let n_j be the number of x_i in the unit lattice square Δ_j; then there exists c such that

$$\left\| \prod_i \psi(x_i) \right\|_p \leqslant c^{\sum n_j} \prod_j (n_j!) \tag{5.21}$$

Here c goes to ∞ as λ goes to 0.

The factorials in (5.21) can be again estimated by the "exponential pinning lemma," Lemma 9.11 of Ref. 2: Given $c' > 0$ and q, there exists c such that

$$\prod_\alpha (n_\alpha!)^q \leqslant c^{\sum n_\alpha} \exp(c'd) \tag{5.22}$$

$$\prod_a (N_\alpha!)^q \leqslant c^{|X|} \exp[c'(L_0 + d)] \tag{5.23}$$

Here N_α is the number of factors of distinguished ε's in κ'' that are localized in Δ_α. Inequalities (5.22) and (5.23) can be proved as in Ref. 2, with a slight change of constants. We note that our $\mathscr{A}$ is different from the $\mathscr{A}$ in Ref. 2, but that the number of distinguished ε's localized in Δ_α is at most $N_\alpha + 1$. Using the bound $(N_\alpha + 1)! \leqslant 2(N_\alpha!)^2$, we may apply (5.22) and (5.23) to estimate factorials in (5.21).

The result of these estimates is that (5.19) can be bounded by $\|e^{\gamma B}\|_{p_3}$ $Q(\beta, p_3)$, with $p_3 > p$. Here $Q(\beta, p_3)$ is the supremum over compatible parameters of the form

$$\exp(c_2' \,|X|)\, c^{\sum w_\alpha} f_q \beta^{T/6} \int |\bar{J}|\, \Pi(|F_2'|)\, \Pi(|g-h|)$$

Here, $T/6$ is the power of β obtained from the above estimates. The factors $\exp(c_1 d)$, $\exp(\delta L_0)$, and $\exp(rO_0)$ are included in J; constants c_1, δ have been increased. Note that c is independent of λ, while $c_2' \to \infty$ as $\lambda \to 0$. The factor f_q is defined by $\Pi_\alpha(n_\alpha!)^{-q}(N_\alpha!)^{-q}$.

We shall show that $Q(\beta, p_3)$ satisfies our estimates (i) and (ii) in Proposition 5.6.

Step 4. We shall show that

$$e^{-c_3 F_1/4}\, e^{c_2 |X|}\, \beta^{|X \setminus (\Sigma^\wedge \cup X_1)|/6} \tag{5.24}$$

goes to zero uniformly in X when $|X| > |X_1|$, and is bounded by $\exp(c_2\,|X_1|)$ when $X = X_1$, as β goes to zero.

Note that $|\Sigma^\wedge|$ is bounded by a constant c times the number of segments of size L of discontinuities of h, where $c = c(L, L')$. By (5.5), we then obtain

$$\exp(-c_1 F) \leqslant \exp(-c_1 c \beta^{-1/2}\, |\Sigma^\wedge|) \tag{5.25}$$

For any nonnegative integer q, we have

$$\exp(-c_1 c\beta^{-1/2}) \leqslant (c_1 c)^{-q} \beta^{q/2} q!$$

Therefore, for any nonnegative integer q,

$$\exp(-c_1 F_1) \leqslant [(c_1 c)^{-q} \beta^{q/2} q!]^{|\Sigma^\wedge|} \tag{5.26}$$

Let $q = 1$ in (5.26); we obtain

$$e^{-c_3 F_1/4} \leqslant \beta^{2|\Sigma^\wedge|/6} \tag{5.27}$$

if β is sufficiently small according to c_3, L, L'. By (5.27), then (5.24) is bounded by

$$e^{c_2|X|} \beta^{[|X \backslash (\Sigma^\wedge \cup X_1)| + 2|\Sigma^\wedge|]/6} \leqslant e^{c_2|X|} \beta^{|X \backslash X_1|/6}$$

the right-hand side of which goes to zero uniformly in X for $|X| > |X_1|$, and is bounded by $\exp(c_2 |X_1|)$ when $X = X_1$, as β goes to zero.

Step 5. We shall prove that there exists $c_5 > 0$ such that, if β is sufficiently small, then

$$e^{-c_5|X|} e^{-c_3 F_1/4} \beta^{-1/6|X \backslash (\Sigma^\wedge \cup X_1)|} \beta^{T/6} \beta^{-3|X_0|/6} \int |\bar{J}| \tag{5.28}$$

is bounded by $\|\zeta\|$, for all X.

Let J be a product of ω many ρ^{η^Δ}'s and possibly some C's with $\exp(c_1 d)$, $\exp(\delta L_0)$, and $\exp(r O_0)$ induced. In view of Theorems 3.1 and 3.2, we must produce enough power of β to compensate the factors β^{-1} that come from estimating the integrals of the ρ^{η^Δ}.

We shall first show that

$$T \geqslant |X \backslash (\Sigma^\wedge \cup X_1)| + 6\omega + 3\,|X_0| \tag{5.29}$$

if we allow (5.28) to include $c^{|X|} \exp(c' L_0)$.

From our expansion, for each $\Delta \in X \backslash (\Sigma^\wedge \cup X_1)$, either a distinguished ε, a derivative of ε, or a differentiation $[\partial/\partial\psi(x)]$ is localized in Δ. Each case produces at lest a factor of $\beta^{1/6}$. For an $\mathscr{E}$ such that $t \geqslant 3$, if there are two more distinguished or differentiated $\mathscr{E}$'s, then we have a factor of β. For an $\mathscr{E}$ such that $t = 2$, if there is one more $\bar{\varepsilon}$ or a derivative of $\bar{\varepsilon}$, we have a factor β. Let S, $|S| = s$, be the set of $\mathscr{E}$'s where the above choice is impossible. For each $\mathscr{E}_i \in S$, there exists at least one ε, which we call ε_i, in $\mathscr{E}_i$ such that ε_i is not differentiated and localized in class B. To bound 6ω in the right side of (5.29), it is sufficient to prove that, for any c', $c_3 > 0$, there exists c such that

$$\exp(-c_3 F_1/8) \leqslant c^{|X|} \exp(2c' L_0)\, \beta^{3s/2} \tag{5.30}$$

To prove (5.30), we note that S contains at most two $\mathscr{E}$'s from each $\kappa''(j)$. We write $S = S_1 \cup S_2$, where each S_i contains at most one $\mathscr{E}$ from each $\kappa''(j)$. We shall consider S_1 only, as S_2 can be treated in the same way. Let $|S_1| = s_1$. Let $q_{1\Delta}$ = the number of ε_i localized in Δ, where $\mathscr{E}_i \in S_1$, $\mathscr{E}_i$ is from $\kappa''(i)$, and $\Delta \in Y_{i+1}$. We put $q_{2\Delta}$ = the number of ε_i localized in Δ, where $\mathscr{E}_i \in S_1$, $\mathscr{E}_i$ is from $\kappa''(i)$, and $\Delta \notin Y_{i+1}$.

If $\mathscr{E}_i(a)$ contributes to $q_{2\Delta}$, then (a) must contain Δ and a lattice square in Y_{i+1}. Recall that Y_{i+1} are disjoint. Therefore

$$\sum_i L_{\eta^A}(a) \geqslant \sum d(\Delta, \Delta_j) \tag{5.31}$$

where the left-hand side summations are over all i such that $\mathscr{E}_i(a)$ contributes to $q_{2\Delta}$; in the right-hand side summation Δ_j, $j = 1,..., q_{2\Delta}$, are chosen to be distinct and as close to Δ as possible.

We sum over all Δ such that $q_{2\Delta} \neq 0$, to obtain

$$\sum_\Delta \sum_j d(\Delta, \Delta_j) \leqslant \sum L_{\eta^A}(a) \tag{5.32}$$

Here the right-hand side summation is over all $\mathscr{E}(a) \in S_1$, the left-hand side summation is over all Δ such that $q_{2\Delta} \neq 0$, $j = 1,..., q_{2\Delta}$, and Δ_j are distinct and as close to Δ as possible.

By exponential pinning [see, e.g., (A2.2) in Ref. 2], for any $c' > 0$ there exists c such that

$$\prod_\Delta (3q_{2\Delta}!) \leqslant c^{|X|} \exp\left[c' \sum_\Delta \sum_j d(\Delta, \Delta_j)\right] \leqslant c^{|X|} \exp(c' L_0) \tag{5.33}$$

By (5.26), there exists $c > 0$ such that, for $i = 1, 2$,

$$e^{-c_3 F_1/32} \leqslant \prod_\Delta c^{q_{i\Delta}} \beta^{3q_{i\Delta}/2} (3q_{i\Delta})! \tag{5.34}$$

where the product is taken over all Δ such that $q_{i\Delta} \neq 0$.

We note that $q_{1\Delta} \leqslant 1$. By (5.33) and (5.34), we obtain that, for any $c' > 0$, there exists c such that

$$e^{-c_3 F_1/16} \leqslant c^{|X|} e^{c' L_0} \beta^{3s_1/2} \tag{5.35}$$

We apply the same argument to S_2; then we have proved (5.30), where c has been increased, $c = c(L', L, c')$.

For $3\,|X_0|$ in the right side of (5.29), we shall separate X_0 into an union of X_{01} and X_{02}, where X_{01} is the union of squares in class A, and X_{02} is the union of squares in class B. By (5.26),

$$e^{-c_3 F_1/8} \leqslant c^{|X_{02}|} \beta^{3|X_{02}|/6} \tag{5.36}$$

For any square Δ in X_{01}, there exists an ε from $\mathscr{A}$ localized in Δ. Whether ε is differentiated or not, we obtain at least a power of $\beta^{1/2}$. Combining this with (5.36), we obtain a power of $3\,|X_0|$.

The factor $\exp(2c'L_0)$ is again absorbed in J. To estimate $\int |\bar{J}|$, we note that the supremum of C exists; therefore, we may drop all C's in $\bar{J}$. We obtain a product of $\int \rho^{\eta^A}$'s and $\|\zeta\|$ as an upper bound for $\int |\bar{J}|$, which may be estimated by Theorems 3.1 and 3.2.

To show Lemma 6.1, we would like to estimate $\int |\bar{J}|$ in terms of $\|C\|$, which is again bounded by $\sup_x \int C(x, y)\, dy$, whenever a factor C appears in $\bar{J}$. If $\bar{J}$ contains at least one C, we then drop all C's except one. Then the factor that includes C must be one of the following six types:

$$\int \rho(x_1, \ldots, x_t)\, C(x_i, y_j)\, \rho(y_1, \ldots, y_j, \ldots, y_s)$$

$$\int \rho(x_1, \ldots, x_t)\, C(x_i, y)$$

$$\int C(x, y)$$

$$\int \rho(x_1, \ldots, x_t)\, C(x_i, y_j)\, \zeta(y_1, \ldots, y_{w_1+w_2})$$

$$\int C(y_i, y_j)\, \zeta(y_1, \ldots, y_{w_1+w_2})$$

$$\int \zeta(y_1, \ldots, y_{w_1+w_2})\, C(y_i, y)$$

Here, each ρ has a certain superscript η^A, and integrations in x's, y's are over certain lattice squares. We can estimate the above integrals by using $\|C\|$ and possibly some of the following factors:

$$\int |\rho(x_1, \ldots, x_t)|, \qquad \|\zeta\| \tag{5.37}$$

$$\sup \int |\rho(y_1, \ldots, y_s)|\, dy_1 \cdots dy_{j-1}\, dy_{j+1} \cdots dy_s \tag{5.38}$$

$$\int_{\Delta_i} dy_i \sup_{y_j \in \Delta_j} \int \cdots \int |\zeta(y_1, \ldots, y_{w_1+w_2})| \prod_{s \neq i,j} dy_s \tag{5.38$'$}$$

where, in (5.38), the supremum is taken y_j in R^2; in (5.38′), Δ_i and Δ_j have disjoint interior. We may estimate the first factor of (5.37) and (5.38) by using Theorems 3.1 and 3.2. Fro Lemma 6.1, $\zeta=\rho$, we may estimate (5.38′) and $\|\zeta\|$ by (3.30) and (3.31).

Combining the above estimates and (5.29), we find that if we choose c_5 to be sufficiently large, then (5.28) is bounded by $\|\zeta\|$, for all X, as β goes to zero.

Step 6. We shall prove that there exist q and $c_5>0$ such that, for β sufficiently small, then

$$\exp(-c_5\,|X|-c_3F_1/2)\,f_q\int \Pi(|g-h|)\,\Pi(|F_2'|) \tag{5.39}$$

is bounded by 1 for all X.

Let Δ be a unit lattice square. We denote by $\sum_f$ the summation over all internal lines of the lattice squares of size L in Δ. By (7.19) Ref. 2 we have

$$\int_\Delta |g-h|^2 \leqslant L^2 \sum_f |\delta h(f)|^2 \tag{5.40}$$

The integral $\int_\Delta |F_2'|$ can be bounded by $\{\int_\Delta |F_2'|^2\}^{1/2}$. By estimates (9.414)–(9.419) in Ref. 2, there exists c such that

$$\left(\int_\Delta |F_2'|^2\right)^{1/2} \leqslant c\left[\sum_f |\delta h(f)|^2\right]^{1/2} \tag{5.41}$$

We may drop 1/2 from the right side of (5.41) because $\sum_f |\delta h(f)|^2$ is either 0 or greater than 1 if β is sufficiently small.

The total factor of $|g-h|$ and $|F_2'|$ localized in Δ is bounded by n_Δ. By (5.5), (5.40), and (5.41), (5.39) is bounded by

$$\exp(-c_5\,|X|)\exp\left[-c_3c'\sum|\delta h(f)|^2\right] f_q \prod_\Delta \left[c\sum_f |\delta h(f)|^2\right]^{n_\Delta} \tag{5.42}$$

If we choose c_5 to be sufficiently large, then (5.42) is bounded by 1, when β goes to zero.

Step 7. By steps 4, 5, and 6, there exists $c_B''>0$ such that $\exp(-c_1F_1)Q$ is bounded by

$$\|\zeta\|\; c_3^{\Sigma w_\alpha}\exp(c_B''\,|X_1|)\,\beta^{|X_0|/2} \tag{5.43}$$

when $X=X_1$, for β sufficiently small. Here c_3 is independent of λ, while $c_A\to\infty$ as $\lambda\to 0$. When $|X|>|X_1|$, $\exp(-c_1F_1+c_2\,|X|)Q$ is equal to $\|\zeta\|\;c(\beta)\,c_3^{\Sigma w_\alpha}$, where $c(\beta)$ goes to zero as β goes to zero. This completes the proof of Proposition 5.6.

6. PROOF OF THEOREM 2.1

6.1. The following lemmas are analogous to statements in Section 9.9 of Ref. 2.

Lemma 6.1. Let $\mathscr{A}$ be of the form (5.1) with $\zeta = \rho_{e_1,\dots,e_n}(x_1,\dots,x_n)$. Suppose λ is chosen sufficiently small and fixed to be nonzero. If β is sufficiently small according to λ, then

$$\lim_{\Lambda \nearrow R^2} \frac{1}{Z}\frac{1}{Z_0} I(\mathscr{A}) \quad \text{exists}$$

The proof of Lemma 6.1 is based on the convergent expansion (4.14) and the Kirkwood–Salsburg equations. Once we have proved convergence of the cluster expansion (Theorem 5.1), the rest of the proof follows from the same arguments as in Appendix 4 of Ref. 2. Using the "doubling the measure" argument (see, e.g., Refs. 1 and 2), we also obtain the following lemma.

Let Δ_x, Δ_y be unit lattice squares containing x, y, respectively. Let a's be lattice squares of size $\bar{l}_D$. We consider the following observables. Let

$$\begin{aligned}\mathscr{A} &= \mathscr{A}(\Delta_x, e_1,\dots, e_n, a_2,\dots, a_n)\\ &= \int_{\Delta_x} dx_1 \exp[i\beta^{1/2}\phi(x_1)e_1]\\ &\quad \times \prod_{j=2}^{n} \int_{a_j} \varepsilon_{e_j}(x_j)\, \rho_{e_1,\dots,e_n}(x_1,\dots,x_n)\, dx_2 \cdots dx_n \end{aligned} \tag{6.1}$$

$$\begin{aligned}\mathscr{C} &= \int_{\Delta_x} dx_1 \int_{\Delta_y} dx_2 \prod_{j=1}^{2} \exp[i\beta^{1/2}\phi(x_j)e_j]\\ &\quad \times \prod_{j=3}^{n} \int_{a_j} \varepsilon_{e_j}(x_j)\, \rho_{e_1,\dots,e_n}(x_1,\dots,x_n)\, dx_1 \cdots dx_n \end{aligned} \tag{6.2}$$

Let

$$\mathscr{B} = \mathscr{A}(\Delta_y, e_{n+1},\dots, e_{n+m}, a_{n+2},\dots, a_{n+m})$$

Let X_1, X_2, X_3 be the support of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, respectively. Let $Y_1 = \bigcup_{j=2}^{n} a_j$, $Y_2 = \bigcup_{j=2}^{m} a_{j+n}$, and $Y_3 = \bigcup_{j=3}^{n} a_j$. We shall denote the distance of X_1, X_2 by d.

Lemma 6.2. Let $0 < \delta_1 < 1/2$ be fixed; suppose we choose λ

sufficiently small but nonzero. Then there exist c_3 (independent of λ) and c_Λ such that if β is sufficiently small, then for all Λ we have

$$\begin{aligned}
|\langle \mathscr{A}\mathscr{B}\rangle_\Lambda - \langle \mathscr{A}\rangle_\Lambda \langle \mathscr{B}\rangle_\Lambda|
&\leqslant c_3^{m+n}\{\exp[c_\Lambda(|X_1|+|X_2|)]\}\,\beta^{(|Y_1|+|Y_2|)/2}\\
&\quad\times \exp[-(1-2\delta_1)\,d/\tilde{l}_{\mathrm{D}}]\\
&\quad\times k_n(\Delta_x, a_2,..., a_n)\,k_m(\Delta_y, a_{2+n},..., a_{m+n})
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
|\langle \mathscr{C}\rangle_\Lambda| &\leqslant c_3^n[\exp(c_\Lambda\,|X_3|)]\,\beta^{|Y_3|/2}\\
&\quad\times \int_{\Delta_x}\int_{\Delta_y}\int_{a_3}\cdots\int_{a_n} |\rho_{e_1,...,e_n}(x_1,..., x_n)|\,dx_1\cdots dx_n
\end{aligned} \tag{6.4}$$

Here k_n is defined as in (3.15).

The L^1-norm of ρ in (6.3) and (6.4) is from Theorem 5.1 if we replace $\mathscr{A}$ there by the present $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$. Going through the proof of Theorem 5.1, in Proposition 5.6, we see that we have used the supremum norms of $\exp[i\beta^{1/2}\phi(x)e]$, or ε, or their derivatives with respect to ϕ. Therefore we obtain the factor of the L^1-norm of the ρ's in (6.3) and (6.4).

6.2. We consider the following correlation function of two-point charge densities in Λ:

$$\left\langle \int_{\Delta_x} J(x_1)\,dx_1 \int_{\Delta_y} J(x_2)\,dx_2\right\rangle_\Lambda - \left\langle \int_{\Delta_x} J(x_1)\,dx_1\right\rangle_\Lambda \left\langle \int_{\Delta_y} J(x_2)\,dx_2\right\rangle_\Lambda \tag{6.5}$$

We assume

$$d = d(\Delta_x, \Delta_y) \geqslant \tilde{l}_{\mathrm{D}} \tag{6.6}$$

We apply the sine-Gordon transformation to (6.5); then (6.5) can be written as $\beta^{-1}(\mathrm{I}+\mathrm{II})$ [see, e.g., (9.912) of Ref. 2], where

$$\mathrm{I} = \left\langle \int_{\Delta_x}\int_{\Delta_y} \frac{\partial^2 M(\phi)}{\partial\phi(x_1)\,\partial\phi(x_2)}\,dx_1\,dx_2\right\rangle_\Lambda \tag{6.7}$$

$$\begin{aligned}
\mathrm{II} &= \left\langle \int_{\Delta_x} \frac{\partial M(\phi)}{\partial\phi(x_1)}\,dx_1 \int_{\Delta_y} \frac{\partial M(\phi)}{\partial\phi(x_2)}\,dx_2\right\rangle_\Lambda\\
&\quad - \left\langle \int_{\Delta_x} \frac{\partial M(\phi)}{\partial\varphi(x_1)}\,dx_1\right\rangle_\Lambda \left\langle \int_{\Delta_y} \frac{\partial M(\phi)}{\partial\phi(x_2)}\,dx_2\right\rangle_\Lambda
\end{aligned} \tag{6.8}$$

For any $l' > l_{\mathrm{D}}$, we let $3\delta_1 = (l' - l_{\mathrm{D}})/l'$. If we choose λ so small that $|l_{\mathrm{D}} - \tilde{l}_{\mathrm{D}}|/l' \leqslant \delta_1$, then $(1-2\delta_1)/\tilde{l}_{\mathrm{D}} \geqslant (l')^{-1}$. Therefore, the following lemmas are sufficient for proving Theorem 2.1.

Lemma 6.3. Under the same conditions on δ_1, λ, and β as in Lemma 6.2, there exists a constant c such that

$$|\mathrm{I}| \leqslant c\tilde{z}^2\beta^2 \exp(-d/\tilde{l}_{\mathrm{D}}) \tag{6.9}$$

$$|\mathrm{II}| \leqslant c\tilde{z}^2\beta \exp[-(1-2\delta_1)\, d/\tilde{l}_{\mathrm{D}}] \tag{6.10}$$

Lemma 6.4. Under the same conditions on δ_1, λ, and β as in Lemma 6.2, the infinite-volume limit exists for I, II, respectively.

Proof of Lemma 6.3. By (3.3), $M(\phi)$ is a summation over n and over (a) of $\mathscr{E}(a)$. For each $\mathscr{E}(a)$,

$$\beta^{-1}n! \int [\partial\mathscr{E}(a)/\partial\phi(x_1)]\, dx_1$$

is a summation of at most n terms of the form (6.1), and

$$\beta^{-1}n! \iint [\partial^2\mathscr{E}(a)/\partial\phi(x_1)\, \partial\phi(x_2)]\, dx_1\, dx_2$$

is a summation of at most $n(n-1)$ terms of the form (6.2). Here x_1 and x_2 are integrated over Δ_x, Δ_y, respectively. Let ω be the minimal number of lattice squares of size $\tilde{l}_{\mathrm{D}}$ such that their union covers $\Delta_x \cup \Delta_y$. By (6.4), $|\mathrm{I}|$ is bounded by

$$\beta(n!)^{-1} \sum_{n=2}^{\infty} \sum_{e_1,\ldots,e_n} \sum_{a_3,\ldots,a_n} n(n-1)\, c_3^n\, e^{\omega c_A}$$
$$\times \int_{\Delta_x}\int_{\Delta_y}\int_{a_3}\cdots\int_{a_n} |\rho_{e_1,\ldots,e_n}(x_1,\ldots, x_n)|\, dx_1 \cdots dx_n \tag{6.11}$$

for β sufficiently small.

By Theorem 3.4, (6.11) is bounded by

$$\beta \sum_{n=2}^{\infty} n(n-1)(2c_3)^n[\exp(\omega c_A)]\, c(\beta)\, 2(\kappa'')^n$$
$$\times e^{-1}\beta^{-1}(\|w\|'')^{-2} \sum_{\eta^A} b_{\eta^A} \times \exp[-L_{\eta^A}(\Delta_x, \Delta_y)\, \tilde{l}_{\mathrm{D}}] \tag{6.12}$$

We note that $c(\beta)$ is bounded as β goes to zero. We first choose λ so small that $2\kappa''c_3 < 1$ and we choose β to be sufficiently small according to λ; then (6.12) is bounded by a constant times $\tilde{z}^2\beta^2 \exp(-d/\tilde{l}_{\mathrm{D}})$.

We shall apply (6.3) to estimate II. If β is sufficiently small according to c_A and hence to λ, then $|\mathrm{II}|$ is bounded by

$$\beta \sum_{n\geqslant 1} \sum_{m\geqslant 1} nm \frac{1}{n!}\frac{1}{m!} c_3^{n+m}[\exp(\omega c_A)]\, 2^n 2^m$$
$$\times \sum_{(a)} \sum_{(a')} k_n(\Delta_x, a_2,\ldots, a_n)\, k_m(\Delta_y, a_{2+n},\ldots, a_{m+n})$$
$$\times \exp\frac{-(1-2\delta_1)\,\mathrm{dist}(X_1, X_2)}{\tilde{l}_{\mathrm{D}}} \tag{6.13}$$

By Theorem 3.1, (6.13) is bounded by

$$\beta\left(\exp\frac{-(1-2\delta_1)\,d}{\tilde{l}_{\mathrm{D}}}\right)$$
$$\times\left[\sum_{n\geqslant 1}\frac{1}{(n-1)!}\,c_n(\alpha)(2c_3)^n\sum_{(a)}\sum_{\eta^A}b_{\eta^A}\exp\frac{-2\delta_1 L_{\eta^A}}{\tilde{l}_{\mathrm{D}}}\right]^2 \tag{6.14}$$

Using the estimate of $c_n(\alpha)$ in Theorem 3.1 with $\alpha=1$, we may bound (6.14) by

$$\beta\left(\exp\frac{-(1-2\delta_1)\,d(\Delta_x,\Delta_y)}{\tilde{l}_{\mathrm{D}}}\right)\left\{\sum_{n\geqslant 1}\frac{2\kappa^n n}{e\beta\;\|w\|}(2c_3)^n[c(\delta_1)]^{n-1}\right\}^2 \tag{6.15}$$

Here $c(\delta_1)$ is obtained by (3.7), which is an estimate of summing over (a). If we choose λ so small that $2\kappa c_3 c(\delta_1)<1$, then (6.15) is bounded by a constant times

$$\beta\tilde{z}^2\exp[-(1-2\delta_1)\,d(\Delta_x,\Delta_y)/\tilde{l}_{\mathrm{D}}]$$

Proof of Lemma 6.4. From the proof of Lemma 6.3, we find that I and II are convergent series uniform in Λ. Each term in the series has a limit as $\Lambda\nearrow R^2$, and therefore I and II have limits as $\Lambda\nearrow R^2$.

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